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The fixed-trace β-Hermite ensemble of random matrices and the low temperature distribution of the determinant of an $N \times N$ β-Hermite matrix

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Abstract

The β-Hermite ensemble (β-HE) of tridiagonal $N \times N$ random matrices of Dumitriu and Edelman (2002 *J. Math. Phys.* **43** 5830) is a continuum of ensembles in which β, the reciprocal of the temperature in the 2D electrostatic interpretation of the eigenvalue characteristics, can take any value. The eigenvalue distributions coincide with those of the classical Gaussian ensembles (GOE, GUE, GSE) for $\beta = 1, 2, 4$. A fixed-trace β-Hermite ensemble (β-FTHE) is defined from the β-HE and is used to extend the spherical ensembles of classical symmetries to β-spherical ensembles. At low temperature, when $\beta \to \infty$, for a fixed value of N, the asymptotic distributions of reduced determinants $D_{N,\beta}$ of random $N \times N$ β-H and β-FTH matrices are shown to be standard Gaussians. Accordingly, the fluctuations of the potential at the origin, $-\ln|D_{N,\beta}|$, have a generalized Gumbel distribution at low temperature. For large N and large β, a $\ln(N)$ variance results from the strongly correlated fluctuations of eigenvalues around their equilibrium positions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

1.1. General

Introduced by Wigner in physics [1], random matrix theory (RMT) continues to be of considerable interest in branches as different as quantum chaology, for investigating growth models or in econophysics to quote just a few [1–7]. Many characteristics of the distributions of eigenvalues of $N \times N$ random matrices from the three fundamental Gaussian ensembles, the GXE's where X = O, U, S means orthogonal, unitary and symplectic respectively, are known both exactly at finite N and asymptotically at large N for which local statistics are often universally distributed once properly scaled. Matrices are real symmetric for the GOE, Hermitian for the GUE and quaternion self-dual for the GSE. Recent works on the GXE matrices focused for instance on the calculations of the exact global densities and on the oscillatory large-N corrections to the Wigner semi-circle, on the averages of characteristic polynomials and of their ratios and on the use of supersymmetry [8–10]. Further, the scaled m largest eigenvalues of large $N \times N$ random matrices from the GXE's are for instance universally distributed according to Tracy–Widom distributions which enlarge the field covered by extreme value theory [4, 5 and references therein].

The number of distinct real random variables, which are necessary to construct a $N \times N$ GXE matrix, is $N_\rho = N + \beta \frac{N(N-1)}{2}$, with $\beta = 1, 2, 4$ for the GOE, the GUE and the GSE respectively, where $\rho = \frac{\beta}{2}$ will be used hereafter. Further, β is zero for an ensemble of diagonal matrices with identically and independently distributed (iid) Gaussian variables. The joint distribution of eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ of $N \times N$ random matrices from the Gaussian ensembles is [1]

$$\begin{cases} P_{N,\beta}(\lambda_1, \ldots, \lambda_N) = K_{N,\beta} \exp\left(-\dfrac{1}{2\sigma^2}\left[\displaystyle\sum_{k=1}^{N} \lambda_k^2\right]\right) \left[\displaystyle\prod_{1 \leqslant j<k \leqslant N} |\lambda_j - \lambda_k|^\beta\right] \\ \rho = \dfrac{\beta}{2} \qquad K_{N,\beta} = \sigma^{-N_\rho/2}(2\pi)^{-N/2} \displaystyle\prod_{j=1}^{N} \dfrac{\Gamma(1+\rho)}{\Gamma(1+j\rho)} \end{cases} \tag{1}$$

where $K_{N,\beta}$ is the reciprocal of the Mehta integral [1, p 354]. The N_ρ distinct elements of the GXE matrices are recalled to be independently distributed according to Gaussian distributions with zero means and variances $\sigma_{ij}^2 = \sigma^2\left(\frac{1+\delta_{ij}}{2}\right)$ for $\beta = 0, 1, 2$, denoted hereafter by $N\left(0, \sigma_{ij}^2\right)$.

1.2. The β-Hermite ensemble of tridiagonal random matrices and the Gaussian log-gas

The properties of eigenvalues of random matrix ensembles can be interpreted in 2D from the equilibrium characteristics of a gas of N identical point charges on a line [1], often referred to as a log-gas [8], which interact via a logarithmic Coulomb potential and are confined by an external potential. The external potential is harmonic in the case of Gaussian ensembles as seen by rewriting equation (1) as a Boltzmann factor at a temperature $1/\beta$ which depends only on the symmetry of the considered ensemble. As β has no reason to be restricted to some integer values in this electrostatic interpretation, it was desirable to extend the previous Gaussian ensembles so as to let β take any positive value. Dumitriu and Edelman [11] found an elegant solution to the latter problem, namely the β-Hermite (β-H) ensemble of tridiagonal random matrices, defined in section 2, whose density of eigenvalues is given by equation (1) whatever β [11–15]. The use of β-H matrices results, among others, in an unrivalled speed up and efficiency of numerical simulations of all characteristics of the eigenvalue distributions of large random matrices (section 2.1). Extensions of other classical ensembles were similarly performed, for instance for the β-Laguerre ensemble [11], for the β-Jacobi ensemble and for β-circular ensembles [16–18].

The β-HE also facilitates the study of the distribution of the determinant D_N of $N \times N$ matrices. The probability densities of random determinants is too a question of physical interest [19–25]. Within the log-gas interpretation, the potential at the origin is $-\ln|D_N|$ which is a linear statistic as are physical quantities expressed as sums $F = \sum_{k=1}^{N} f(\lambda_k)$ over the eigenvalues λ_k of a random matrix. General arguments and proofs predict that the distribution of any linear statistic is Gaussian and independent of the confining potential in the scaled asymptotic limit $N \to \infty$ provided its fluctuation is finite [3, 26, 27]. This leads quite naturally to an asymptotic lognormal distribution of the determinant, when $N \to \infty$ for a fixed value of β, as shown for the GXE (X = O,U). The lognormal distribution is a typical and robust asymptotic distribution of products of random variables which holds for rather loose conditions on these variables. However, the standard-deviation of the Gaussian distribution of $-\ln|D_N|$ is not here proportional to $\sqrt{N}$ as usual, for instance when D_N is a product of iid variables, but to $\sqrt{\ln N}$ as a consequence of the repulsion between eigenvalues. The study of the determinant distribution is thus a way to probe the global fluctuation properties of the spectrum as stressed for linear statistics [26, 27].

1.3. Aims

The first aim of this paper is to define a fixed-trace β-Hermite (β-FTH) ensemble, a member of a family that was first defined by Rosenzweig and Bronk [28, 29] and bears the same relationship to Gaussian ensembles that the microcanonical ensembles to the canonical ensembles in statistical physics [30]. A fixed-trace ensemble, with a given symmetry, is constrained to have $\mathrm{tr}\left(\mathbf{H}_N \mathbf{H}_N^{+}\right) =$ constant, where the matrix $\mathbf{H}_N^{+}$ is the Hermitian conjugate of $\mathbf{H}_N$ and the constant is taken here as 1 without loss of generality. Fixed-trace ensembles were recently investigated [31–33] in particular in relation to spherical random matrix ensembles [33–37], whose probability densities are solely functions of $\mathrm{tr}\left(\mathbf{H}_N \mathbf{H}_N^{+}\right)$. Further, generalized random matrix ensembles belonging to the latter class were deduced from the use of a maximum entropy principle based on a non-extensive q-entropy [36, 37].

The second aim is to study the asymptotic distribution of the determinant $D_{N,\beta}$ of a β-H matrix at low temperature, that is when $\beta \to \infty$ for a fixed value of N. The potential at the origin, $-\ln|D_{N,\beta}|$, is further a sum of correlated random variables (equation (1)). The asymptotic distributions of sums of correlated random variables are a fundamental question in statistical physics [38–42]. The asymptotic joint distribution of the β-H matrix eigenvalues, when $\beta \to \infty$ for a fixed value of N, is a multivariate Gaussian distribution with a rather complicated covariance matrix [12, 43, 44] (see section 2.2). The asymptotic determinant distribution shall be obtained here, together with its mean and variance, by a simple inductive reasoning which does not rely explicitly on the previous multivariate Gaussian distribution.

The β-Hermite ensemble and the fixed-trace β-Hermite ensemble shall be defined and some of their characteristics discussed in sections 2 and 3, respectively. The low-temperature determinant distributions for both ensembles shall be discussed in section 4.

2. The β-Hermite ensemble of Dumitriu and Edelman [11]

2.1. Definition

Random matrices from the β-HE, with a density of eigenvalues given by equation (1) whatever β, are tridiagonal symmetric real matrices whose elements are distributed in the following way [11]:

$$\mathbf{A}_{N,\beta} = \sigma \mathbf{H}_{N,\beta} = \sigma \begin{bmatrix} H_{11} & H_{12}/\sqrt{2} & 0 & \cdot & 0 \\ H_{12}/\sqrt{2} & H_{22} & H_{23}/\sqrt{2} & 0 & \cdot \\ 0 & H_{23}/\sqrt{2} & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & H_{N-1,N-1} & H_{N-1,N}/\sqrt{2} \\ 0 & \cdot & 0 & H_{N-1,N}/\sqrt{2} & H_{NN} \end{bmatrix}. \quad (2)$$

The $2N-1$ distinct matrix elements, namely $A_{kk} = \sigma H_{kk} (k = 1, \ldots, N)$ and $A_{k,k+1} = \sigma H_{k,k+1}/\sqrt{2}$ $(k = 1, \ldots, N-1)$ are independently but not identically distributed and σ is a scale factor. Every H_{kk} has a $N(0, 1)$ Gaussian distribution while the off-diagonal element $H_{k,k+1}$ $(k = 1, \ldots, N-1)$ has a chi distribution with $k\beta$ degrees of freedom:

$$P_{N,\beta}(x = H_{k,k+1}) = \frac{x^{k\beta-1}\exp\left(-\frac{x^2}{2}\right)}{2^{k\beta/2-1}\Gamma\left(\frac{k\beta}{2}\right)} \qquad (H_{k,k+1} \geqslant 0). \tag{3}$$

It follows immediately that $\operatorname{tr}\left(\mathbf{H}^2_{N,\beta}\right)$, a sum of independent chi-square random variables, has a chi-square distribution with N_ρ degrees of freedom whatever β.

The β-Hermite ensemble is particularly suited for efficient numerical simulations of its various characteristics with computer times essentially independent of β. We performed Monte Carlo calculations in Fortran with a standard laptop computer to simulate β-H matrices (times of 0.03 s and 0.12 s were for instance needed to build and to diagonalize a 200×200 matrix and a 400×400 matrix respectively). Gaussian variables were generated by the polar Box–Muller method [45]. The chi distributions of the non-diagonal elements were generated through gamma distributions. Ghosh *et al* [46] describe Monte Carlo and Langevin methods to generate numerically non-Gaussian ensembles from the 2D Coulomb gas described in section 1.2 with a confining potential V. Charges are moved stochastically on a line until an equilibrium is reached. Their method generates the β-HE when V is chosen to be an harmonic potential. In the latter case, it is preferable however to use the tridiagonal ensemble of Dumitriu and Edelman.

2.2. *Eigenvalue density of a β-HE $N \times N$ tridiagonal matrix when $\beta \to \infty$ [12]*

When $\beta \to \infty$, the off-diagonal element $H_{k,k+1}$ can be written as $\sqrt{k\beta} + \frac{X}{\sqrt{2}}$, where X is a $N(0, 1)$ Gaussian so that $\mathbf{H}_{N,\beta}$ (equation (2)), scaled by $\frac{1}{\sqrt{2N\beta}}$, becomes [12]

$$\begin{aligned}\mathbf{H}^{(s)}_{N,\beta} &= \mathbf{H}_{N,\beta}/\sqrt{2N\beta}\\ &= \frac{1}{\sqrt{2N}}\mathbf{H}^{(a)}_N + \frac{1}{\sqrt{2N\beta}}\begin{bmatrix} Z_1 & Z_{N+1} & 0 & \cdot & 0\\ Z_{N+1} & Z_2 & Z_{N+2} & 0 & \cdot\\ 0 & Z_{N+2} & \cdot & \cdot & 0\\ \cdot & 0 & \cdot & Z_{N-1} & Z_{2N-1}\\ 0 & \cdot & 0 & Z_{2N-1} & Z_N\end{bmatrix}.\end{aligned} \tag{4}$$

where $Z_N = (Z_1, \ldots, Z_N, Z_{N+1}, \ldots, Z_{2N-1})$ is a vector whose components Z_k are standard $N(0, 1)$ Gaussians for $k \leqslant N$ and $N(0, 1/4)$ Gaussians for $k > N$. When $\beta \to \infty$, the ordered eigenvalues $\lambda_{N,i} (1 \leqslant i \leqslant N)$ of $\mathbf{H}^{(s)}_{N,\beta}$ are such that

$$\sqrt{\beta}\left(\lambda_{N,1} - \frac{x_{N,1}}{\sqrt{2N}}, \lambda_{N,2} - \frac{x_{N,2}}{\sqrt{2N}}, \ldots, \lambda_{N,N} - \frac{x_{N,N}}{\sqrt{2N}}\right) \to \frac{\mathbf{G}}{\sqrt{2N}} \tag{5}$$

where the $x_{N,i}$ are the ordered roots of the Hermite polynomial, $H_N(x_{N,i}) = 0$, and $\mathbf{G} = (G_1, G_2, \ldots, G_N)$ is a vector of Gaussian variables whose means are zero. The Hermite polynomial $H_N(\lambda)$ is classically defined by [55, 56]

$$\int_{-\infty}^{+\infty} \exp(-\lambda^2) H_N(\lambda) H_M(\lambda)\, \mathrm{d}\lambda = \sqrt{\pi}2^N N!\delta_{NM} \tag{6}$$

with a positive coefficient of λ^N while $\hat{H}_N(\lambda) = \frac{1}{(\sqrt{\pi}2^N N!)^{1/2}} H_N(\lambda)$ is an orthonormal Hermite polynomial (equation (7)). The elements of the covariance matrix are rather involved and given by [12]

$$\langle G_i G_j \rangle = \frac{\sum_{k=0}^{N-1} \hat{H}_k^2(x_{N,i})\hat{H}_k^2(x_{N,j}) + \sum_{k=0}^{N-2} \hat{H}_{k+1}(x_{N,i})\hat{H}_k(x_{N,i})\hat{H}_{k+1}(x_{N,j})\hat{H}_k(x_{N,j})}{\left(\sum_{k=0}^{N-1} \hat{H}_k^2(x_{N,i})\right)\left(\sum_{K=0}^{N-1} \hat{H}_k^2(x_{N,j})\right)}. \quad (7)$$

The asymptotic density of eigenvalues is finally obtained as a sum of Gaussians [12]. The eigenvalues of the covariance matrix (equation (8)) are $\frac{1}{k}(k = 1, \ldots, N)$, as found too from equation (11) of [43]. The covariance matrix described above is indeed proportional to $-\boldsymbol{C}^{-1}$ (the elements of the matrix $\boldsymbol{C}$ are simple and given by equations (6) and (7) of [43]). Andersen *et al* [43] derived the multivariate Gaussian distribution of the eigenvalues by expanding the logarithm of the multivariate probability (equation (1) with $\sigma^2 = \frac{1}{\beta N}$) in the vicinity of its maximum to describe the normal modes of the eigenvalue spectrum. The most probable fluctuation in the spectrum corresponds to a common shift of all charges, without change in their relative separation, which reflects the spectral rigidity [43]. Andersen *et al* showed that the next most probable mode is a breathing mode.

2.3. Determinant distribution of the β-HE

General expressions for the distribution of the determinant $D_{N,\beta}$ of a $N \times N$ β-H matrix and for its Mellin transform are lacking for an arbitrary value of β. The asymptotic determinant distribution for β fixed and $N \to \infty$ is expected to be lognormal as $\ln|D_{N,\beta}|$ is a linear statistics (section 1.2, figure 5(*a*) with $\beta = 2$). Exact determinant distributions $P(D_{N,\beta})$ were established previously only for Gaussian ensembles, with $\beta = 0, 1, 2$, with N odd when $\beta = 1$ [21, 22].

3. The fixed trace β-Hermite ensemble

3.1. Definition

We define quite naturally the associated fixed-trace β-Hermite ensemble as the ensemble of matrices:

$$\begin{cases} \mathbf{F}_{N,\beta} = \begin{bmatrix} F_{11} & F_{12}/\sqrt{2} & 0 & \cdot & 0 \\ F_{12}/\sqrt{2} & F_{22} & F_{23}/\sqrt{2} & 0 & \cdot \\ 0 & F_{23}/\sqrt{2} & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & F_{N-1,N-1} & F_{N-1,N}/\sqrt{2} \\ 0 & \cdot & 0 & F_{N-1,N}/\sqrt{2} & F_{NN} \end{bmatrix}. \\ F_{ij} = H_{ij}\Big/\sqrt{\mathrm{tr}\left(\mathbf{H}_{N,\beta}^2\right)}, \ \mathrm{tr}\left(\mathbf{F}_{N,\beta}^2\right) = 1 \end{cases} \quad (8)$$

As explained in section 1.3, for a given symmetry, a fixed-trace ensemble is defined by the sole condition that $\mathrm{tr}\left(\mathbf{H}_N \mathbf{H}_N^+\right) =$ constant taken here as 1. For convenience, the matrix elements will be equally denoted hereafter as F_i $(i = 1, \ldots, 2N - 1)$ with

$$\begin{cases} F_i = F_{ii}, & F_{N+i} = F_{i,i+1} \\ -1 \leqslant F_i \leqslant 1 & \text{if} \quad i \leqslant N \quad 0 \leqslant F_i \leqslant 1 \quad \text{if} \quad i > N. \end{cases} \quad (9)$$

The vector $\mathbf{S}_{N,\beta}$ collects the squares of the distinct elements of $\mathbf{H}_{N,\beta}$ (equation (2)):

$$S_k = H_{kk}^2, \quad (k = 1, \ldots, N), \qquad S_{N+k} = H_{k,k+1}^2, \quad (k = 1, \ldots, N-1). \quad (10)$$

They are by definition of the β-HE (section 2.1) chi-square distributed with respective degrees of freedom:

$$\begin{cases} \nu_1 = 1, \nu_2 = 1, \dots, \nu_k = 1, \dots, \nu_N = 1 \\ \nu_{N+1} = \beta, \nu_{N+2} = 2\beta, \dots, \nu_{N+k} = k\beta, \dots, \nu_{2N-1} = (N-1)\beta. \end{cases} \tag{11}$$

The distribution of the vector $\mathbf{V}_{N,\beta} = \mathbf{S}_{N,\beta} / \mathrm{tr}\left(\mathbf{H}_{N,\beta}^2\right)$ is thus a Dirichlet distribution [49, 50]. When S_k, $k = 1, \dots, 2N-1$, are independent random variables distributed as $\chi^2_{\nu_k}$, $k = 1, \dots, 2N-1$, then the joint distribution of $V_j = S_j / \sum_{k=1}^{2N-1} S_k$, $j = 1, \dots, 2N-2$ is [49, 50]

$$\begin{cases} p(V_1, \dots, V_{2N-2}) = \dfrac{\Gamma(N_\rho/2)}{\pi^{N/2}\left[\prod_{k=1}^{N-1}\Gamma(k\rho)\right]} \times \displaystyle\prod_{i=1}^{2N-1} V_i^{\nu_i/2-1} \\ V_j \geqslant 0, j = 1, \dots, 2N-1, \sum_{j=1}^{2N-1} V_j = 1 \end{cases} \tag{12}$$

from which the joint distribution of the nonzero elements F_{ij}, which takes into account the symmetry of the marginal distributions of the diagonal elements ($-1 \leqslant F_{ii} \leqslant 1$, $i = 1, \dots, N$), is finally derived for $N > 2$:

$$\begin{cases} p(F_{11}, \dots, F_{kk}, \dots, F_{NN}, F_{12}, \dots, F_{k,k+1}, \dots, F_{N-2,N-1}) \\ = \dfrac{2^{N-2}\Gamma(N_\rho/2)}{\pi^{N/2}\left[\prod_{k=1}^{N-1}\Gamma(k\beta/2)\right]} \times \left(\displaystyle\prod_{k=1}^{N-2} F_{k,k+1}^{k\beta-1}\right) \times \left(1 - \sum_{k=1}^{N} F_{kk}^2 - \sum_{k=1}^{N-2} F_{k,k+1}^2\right)^{(N-1)\beta/2-1} \\ -1 \leqslant F_{kk} \leqslant 1, k = 1, \dots, N; 0 \leqslant F_{k,k+1} \leqslant 1, k = 1, \dots, N-1. \end{cases} \tag{13}$$

For $N = 2$,

$$p(F_{11}, F_{22}) = \frac{\rho\left(1 - F_{11}^2 - F_{22}^2\right)^{\rho-1}}{\pi}. \tag{14}$$

The joint distribution of any set of m elements ($m \leqslant 2N-2$) $F_{i_1}, \dots, F_{i_k}, \dots, F_{i_m}$, chosen among the $2N-1$ distinct matrix elements of $\mathbf{F}_{N,\beta}$, is simply obtained from the amalgamation property of the Dirichlet distribution [49, p 19]. Denoting by m_d the number of indices $i_k \leqslant N (k = 1, \dots, m)$, we order the m elements so that the m_d diagonal elements are the first. Let the degrees of freedom (equation (12)) associated with the m corresponding components of $\mathbf{S}_{N,\beta}$ (equation (11)), $S_{i_k} (k = 1, \dots, m)$, and their sum be

$$\nu_{i_1}, \nu_{i_2}, \dots, \nu_{i_m}, \nu_{i_{m+1}} = N_\rho - \sum_{k=1}^{m} \nu_{i_k}. \tag{15}$$

Further, the $2N - m - 1$ remaining components of $\mathbf{S}_{N,\beta}$, are summed up in a component $S_{i_{m+1}}$ with $\nu_{i_{m+1}}(m)$ degrees of freedom. Then, the amalgamated vector $\mathbf{V}_{m+1,\beta}$

$$\begin{cases} \mathbf{S}_{m+1,\beta} = \left(S_{i_1}, S_{i_2}, \dots, S_{i_m}, S_{i_{m+1}} = \displaystyle\sum_{i_k=m+1}^{2N-1} S_{i_k}\right) \\ \mathbf{V}_{m+1,\beta} = \mathbf{S}_{m+1,\beta} / \mathrm{tr}\left(\mathbf{H}_{N,\beta}^2\right) \end{cases} \tag{16}$$

has a Dirichlet distribution whose parameters are defined by equation (16). The sought-after joint distribution is finally

$$\begin{cases} p(F_{i_1}, \dots, F_{i_k}, \dots, F_{i_m}) \\ = \dfrac{2^{m-m_d}\Gamma(N_\rho/2)}{\prod_{i_k=1}^{m+1}\Gamma(\nu_{i_k}/2)} \times \left(\displaystyle\prod_{i_k=m_d+1}^{m} F_{i_k}^{\nu_{i_k}-1}\right) \times \left(1 - \sum_{i_k=1}^{m} F_{i_k}^2\right)^{\nu_{i_{m+1}}/2-1} \\ -1 \leqslant F_{i_k} \leqslant 1, i_k = 1, \dots, m_d; 0 \leqslant F_{i_k} \leqslant 1, i_k = m_d + 1, \dots, m \end{cases} \tag{17}$$

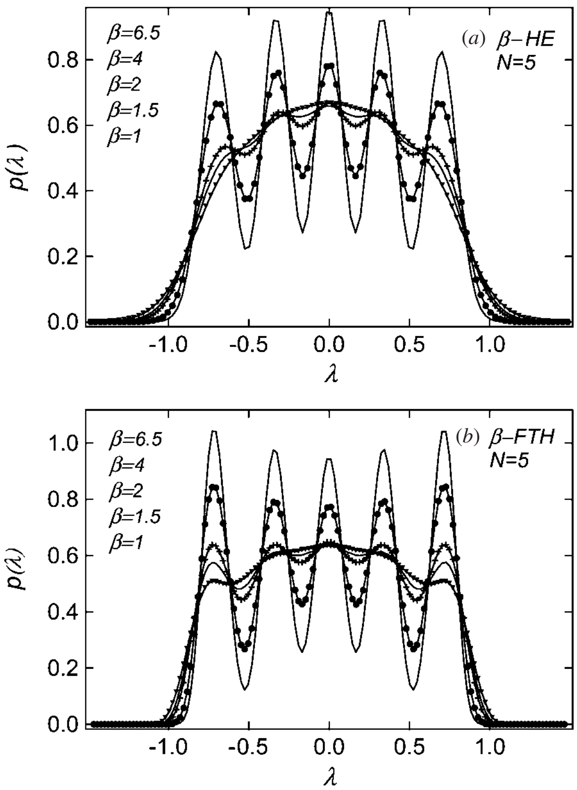

Figure 1. Eigenvalue densities $p(\lambda)$ (*a*) of 5×5 β-Hermite matrices and (*b*) of 5×5 β-fixed trace Hermite matrices, as a function of β from Monte Carlo simulations with 10^7 matrices. In both cases, $\langle\lambda^2\rangle_\beta = 0.25$ and the left peaks correspond, from bottom to top, to $\beta = 1$ (solid line, full triangles), $\beta = 1.5$ (solid line), $\beta = 2$ (solid line, crosses), $\beta = 4$ (solid line, full circles) and $\beta = 6.5$ (solid line).

where the left bracketed factor is taken as 1 if $m_d = m$. Equations (9), (12), (14) and (17) define completely the distribution of the elements of matrices from the fixed-trace β-Hermite ensemble. The marginal distribution of the distinct matrix elements and the distribution of the trace are further discussed in appendix A.

As the F_{ij}'s, whose marginal distributions are given by equations (A.1) and (A.2) (appendix A), are not independently distributed, the simplest method to perform efficient numerical simulations of β-FTH matrices is based on the use of their definition $F_{ij} = H_{ij}\big/\sqrt{\mathrm{tr}\big(\mathbf{H}_{N,\beta}^2\big)}$ (section 2.1). Figure 1 compares some eigenvalue densities of 5×5 β-H and β-FTH matrices when β increases and evidences clearly the smoothing effect originating from equation (19) given below. These densities are easier to understand in the electrostatic interpretation with charges confined more and more strongly around the positions of the zeroes of the Hermite polynomials when the temperature decreases ([1, 12, 43] and section 2.2).

3.2. Eigenvalue densities of β-spherical ensembles

The fixed-trace β-FTHE is then used to define β-spherical (β-S) ensembles of random matrices, whose probability density, $g\big(\mathrm{tr}\big(\mathbf{H}_{N,\beta}^2\big)\big)$, depends only on $\mathrm{tr}\big(\mathbf{H}_{N,\beta}^2\big)$. The characteristics of any

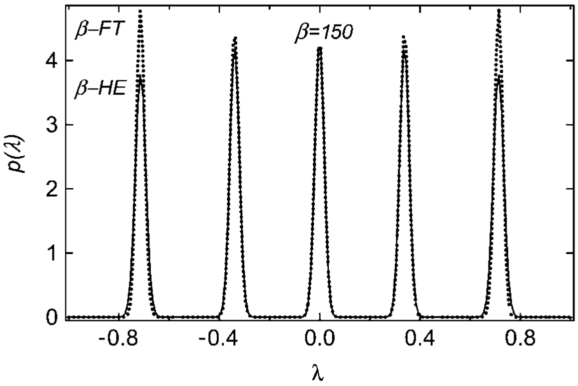

Figure 2. Eigenvalue densities of 5×5 β-fixed trace Hermite matrices (β-FT, dotted line) and of 5×5 β-Hermite matrices (β-HE, solid line) from Monte Carlo simulations with 10^7 matrices; here $\beta = 150$ and $\langle \lambda^2 \rangle = 0.25$.

β-spherical (β-S) ensemble can simply be calculated from those of the β-FTHE by a 1D integration over $r = \sigma\sqrt{\mathrm{tr}\left(\mathbf{H}_{N,\beta}^2\right)}$ as done in [24, 31–37]. The density of eigenvalues of a β-spherical ensemble, $p_{\beta-S,N}(\lambda)$, is related to that of the β-FTHE by equation (2.22) of Delannay and Le Caër [32]. The latter equation, which was solely applied to the cases $\beta = 1$, 2, remains indeed valid whatever β:

$$\begin{cases} p_{\beta\text{-}S,N}(\lambda) = \displaystyle\int_{|\lambda|}^{\infty} \frac{f(r)}{r} p_{\beta\text{-}FTH,N}\left(\frac{\lambda}{r}\right) \mathrm{d}r \\ f(r) \propto r^{N_\rho - 1} g(r^2) \end{cases} \tag{18}$$

where $p_{\beta\text{-}FTH,N}(\lambda)$ is the eigenvalue density of the β-FTHE with $\mathrm{tr}\left(\mathbf{H}_{N,\beta}^2\right) = 1$. When the considered spherical ensemble is the Gaussian β-HE, the chi distribution with N_ρ degrees of freedom of r/σ yields

$$\begin{cases} p_{\beta\text{-}H,N}(\lambda) = C_{N,\beta} \displaystyle\int_{|\lambda|}^{\infty} r^{N_\rho - 2} \exp\left(-\frac{r^2}{2\sigma^2}\right) p_{\beta\text{-}FTH,N}\left(\frac{\lambda}{r}\right) \mathrm{d}r \\ C_{N,\beta} = \dfrac{1}{2^{N_\rho/2-1}\sigma^{N_\rho}\Gamma\left(\frac{N_\rho}{2}\right)}. \end{cases} \tag{19}$$

The density of the β-FTHE might in principle be obtained from that of the β-HE via an inverse one-dimensional Laplace transform as described by equations (3.7) and (3.9) of [32] for $\beta = 1, 2$. The second moment of the β-H eigenvalue distribution is directly found from the chi-square distribution of $z = \mathrm{tr}\left(\mathbf{H}_{N,\beta}^2\right)$ to be $\langle \lambda^2 \rangle_{\beta-H} = \sigma^2\left(1 + \frac{(N-1)\beta}{2}\right)$ from which a thermodynamic limit is obtained when σ scales as $\sigma \propto \frac{1}{\sqrt{\beta N}}$. As the ratio $\frac{\sqrt{\langle (z-\langle z\rangle)^2\rangle}}{\langle z\rangle} \propto \frac{1}{N\sqrt{\beta}}$ tends to zero for large βN^2, the β-HE tends then to the β-FTHE (figure 2) with a radius of the associated sphere $r = \sigma\sqrt{\mathrm{tr}\left(\mathbf{H}_{N,\beta}^2\right)} \propto \sqrt{N}$. When $\beta \to \infty$ for a fixed value of N, the eigenvalue distribution of the β-FTHE is multivariate normal as is that of the β-HE (section 2.2). A relation between the determinant densities of β-H and β-FTH matrices is further given in section 4. Some supplementary characteristics of 2×2 β-H and β-FTH matrices are given in appendix B.

4. Determinant distributions

The probability density $P(D_{N,\beta})$ of the determinant $D_{N,\beta}$ of $N \times N$ β-Hermite matrices is symmetric when $N = 2p + 1$. Indeed, the odd moments, calculated from equation (1) verify

$$\langle(\lambda_1\lambda_2\cdots\lambda_{2p+1})^{2q+1}\rangle = (-1)^{(2p+1)(2q+1)} \qquad \langle(\lambda_1\lambda_2\cdots\lambda_{2p+1})^{2q+1}\rangle = 0 \tag{20}$$

whatever β as shown by changing λ_j into $-\lambda_j$ $(j = 1, \ldots, 2p+1)$. When the mean of a random variable $\langle X\rangle$ differs from zero, the central moments about the mean will be denoted hereafter by $\langle X^k\rangle_c = \langle(X - \langle X\rangle)^k\rangle$.

4.1. β-Hermite ensemble

At low temperature, $\beta \to \infty$, for N fixed, the eigenvalue distribution tends to a multivariate Gaussian distribution (equation (5)) and the determinant distribution becomes that of a product of correlated Gaussian variables (equation (7)). A product of Gaussian variables may, in some conditions, be approximately Gaussian as discussed in appendix C for independent variables. However, these conditions and the mean and variance are harder to derive for the correlated Gaussian variables considered here as further the covariance matrix is rather complex (equation (7)). We therefore derive the sought-after distribution by induction without relying on the previous multivariate distribution.

4.1.1. Low-temperature behaviour of $D_{N,\beta}$ when $\beta \to \infty$ for fixed N. For $N \geqslant 3$ and $\sigma = 1$, equation (2) shows that

$$D_{N,\beta} = H_{NN}D_{N-1,\beta} - \frac{H_{N-1,N}^2 D_{N-2,\beta}}{2}. \tag{21}$$

All moments of the determinant of a β-H matrix are therefore integer coefficients polynomials in $\rho = \beta/2$ [11]. This stems from the moments of an $N(0, 1)$ distribution which are integers and those of the involved chi-square distributions which are polynomials in ρ with integer coefficients:

$$\left\langle\left(\frac{\chi_{k\beta}^2}{2}\right)^n\right\rangle = k\rho(k\rho+1)\cdots(k\rho+n-1). \tag{22}$$

For large ρ, it suffices thus to obtain their term of highest degree. As shown earlier (equation (20)) $\left\langle D_{2p+1,\beta}^{2q+1}\right\rangle = 0$. In equation (21), $D_{N-1,\beta}$ and $D_{N-2,\beta}$ are not independent but $D_{N-1,\beta}$ is independent of H_{NN} and $D_{N-2,\beta}$ is independent of $H_{N-1,N}$. Then

$$\langle D_{2p,\beta}\rangle = -\rho(2p-1)\langle D_{2p-2,\beta}\rangle = (-1)^p(2p-1)!!\rho^p. \tag{23}$$

The latter result is alternatively obtained from the product of the roots of the Hermite polynomial $H_{2p}(x)$. The leading terms of the variance of $D_{N,\beta}$ are derived from the previous equations. Equation (21) yields

$$\left\langle D_{N,\beta}^2\right\rangle = \left\langle D_{N-1,\beta}^2\right\rangle + \rho(N-1)(1+\rho(N-1))\left\langle D_{N-2,\beta}^2\right\rangle \tag{24}$$

equation (24) and the exact values of $\left\langle D_{2,\beta}^2\right\rangle = 1 + \rho + \rho^2$ and of $\left\langle D_{3,\beta}^2\right\rangle = 1 + 3\rho + 5\rho^2$ (appendices C and D) show that $\left\langle D_{N,\beta}^2\right\rangle$ is a polynomial of the form

$$\left\langle D_{N,\beta}^2\right\rangle = 1 + \sum_{i=1}^{2\lfloor N/2\rfloor} a_i(N)\rho^i. \tag{25}$$

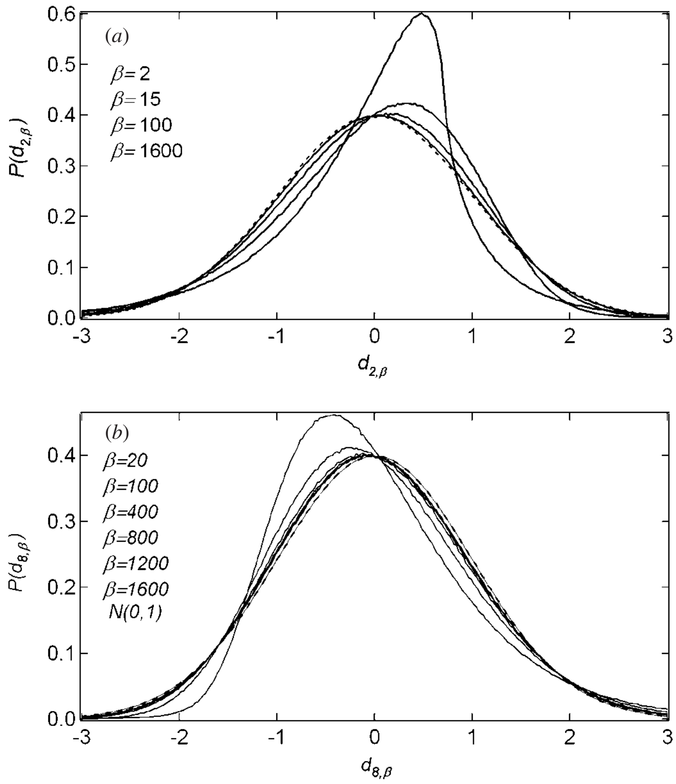

Figure 3. Distribution of the scaled and centred determinant of $N \times N$ β-Hermite matrices as a function of β from Monte Carlo simulations with 10^7 matrices (dotted lines = $N(0, 1)$ distribution, $\frac{1}{\sqrt{2\pi}} \exp(-\frac{d_{N,\beta}^2}{2})$): (*a*) $d_{2,\beta} = (D_{2,\beta} + \rho)/\sqrt{1+\rho}$; β decreases from top to bottom at $d_{2,\beta} \approx 0.5$; (*b*) $d_{8,\beta} = \{D_{8,\beta} - \langle D_{8,\beta}\rangle\}/\sqrt{\langle D_{8,\beta}^2\rangle_c}$; β decreases from top to bottom at $d_{8,\beta} \approx -0.5$.

When N is even, it is then straightforward to find that

$$\left\langle D_{2p,\beta}^2 \right\rangle_c = ((2p-1)!!)^2 \left\{ \sum_{k=0}^{p-1} \frac{1}{2k+1} \right\} \rho^{2p-1} + \cdots. \tag{26}$$

When N is odd, the asymptotic variance is obtained as

$$\left\langle D_{2p+1,\beta}^2 \right\rangle = 4^p (p!)^2 \left\{ \sum_{k=0}^{p} \binom{2k}{k}^2 \Big/ 2^{4k} \right\} \rho^{2p} + \cdots. \tag{27}$$

We define

$$S_{2p}^2 = \sum_{k=0}^{p-1} \frac{1}{2k+1} \qquad S_{2p+1}^2 = \sum_{k=0}^{p} \binom{2k}{k}^2 \Big/ 2^{4k}. \tag{28}$$

A reduced determinant is finally obtained for any value of σ and of N as

$$d_{N,\beta} = \frac{D_{N,\beta} - \langle D_{N,\beta}\rangle}{(N-1)!!\sigma^N \rho^{(N-1)/2} S_N}. \tag{29}$$

For large ρ, $d_{2,\beta}$ (figure 3(*a*)) and $d_{3,\beta}$ are proven to be asymptotically $N(0, 1)$ Gaussians in appendices C and D. To simplify the notations in the derivation by induction of the low temperature distribution of $d_{N,\beta}$ for fixed N and for large ρ, we define $y_N = d_{N,\beta}$ for $N \geqslant 3$ and $\sigma = 1$ and we drop β when possible. Depending on the parity of N, the averages of y_N are $\langle y_{2p+1}\rangle = 0$ and $\langle y_{2p}\rangle = (-1)^p\sqrt{\rho}/S_{2p}$ respectively. The recurrence relation (equation (21)) reads then

$$y_N = \varepsilon_N X_1 \frac{y_{N-1}}{\sqrt{\rho}} - \alpha_N \left(\frac{\chi^2_{(N-1)\beta}}{2}\right) \times \frac{y_{N-2}}{\rho(N-1)} - \langle d_N\rangle \tag{30}$$

where X_1, $\chi^2_{(N-1)\beta}$ have respectively an $N(0,1)$ Gaussian and a chi-square distribution with $(N-1)\beta$ degrees of freedom and

$$\varepsilon_N = \left(\frac{S_{N-1}}{S_N}\right) \times \left(\frac{(N-2)!!}{(N-1)!!}\right) \qquad \alpha_N = \frac{S_{N-2}}{S_N}. \tag{31}$$

Assuming that y_{N-1}, y_{N-2} are asymptotically $N(0,1)$ Gaussians but nothing about their bivariate distribution $P(y_{N-1}, y_{N-2})$ as these two variables are correlated, we calculate the asymptotic distribution of y_N. For that purpose, we derive first the asymptotic characteristic function of y_N, namely $\lim_{\beta\to\infty}\langle\exp(ity_N)\rangle$. From equation (30) we get

$$\begin{aligned}\Phi_N(t) = &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(-\frac{t^2\varepsilon_N^2 y_{N-1}^2}{2\rho}\right)\left(1 + \mathrm{i}t\frac{\alpha_N y_{N-2}}{\rho(N-1)}\right)^{-\rho(N-1)} \\ &\times \exp(-\mathrm{i}t\langle y_N\rangle) P(y_{N-1}, y_{N-2})\,\mathrm{d}y_{N-1}\,\mathrm{d}y_{N-2}.\end{aligned} \tag{32}$$

When N is even: $N = 2p$. The first exponential in the integrand of equation (32) tends to 1 when $\beta \to \infty$ as $y_{N-1}^2 \sim 1$. Similarly, $y_{N-2}^2 \sim \frac{\rho}{S_{N-2}}$. Developing the second factor in the integrand equation (32), with $z_{N-2} = y_{N-2} + (-1)^{N/2}\sqrt{\rho}/S_{N-2}$, where z_{N-2} is $N(0,1)$, we obtain

$$\Phi_N(t) = \exp\left(-\frac{S_{N-2}^2 t^2}{2S_N^2} - \frac{t^2}{2(N-1)S_N^2}\right) = \exp\left(-\frac{t^2}{2}\right). \tag{33}$$

When $\beta \to \infty$ with N fixed, the low-temperature distribution of $d_{2p,\beta} = \frac{D_{2p,\beta} - \langle D_{2p,\beta}\rangle}{(2p-1)!!\rho^{(2p-1)/2}S_{2p}}$ is thus an $N(0,1)$ Gaussian if the asymptotic distributions of $d_{2p-2,\beta}$ and that of $d_{2p-1,\beta}$ are $N(0,1)$ Gaussians.

When N is odd: $N = 2p + 1$. As done previously, we replace y_{N-1}^2 in equation (32) by $\frac{\rho}{S_{N-1}^2}$. Developing the second term gives $\exp\left(-\mathrm{i}t\alpha_N y_{N-2} - \frac{t^2\alpha_N^2 y_{N-2}^2}{2\rho(N-1)}\right)$. Replacing y_{N-2}^2 by 1 yields finally

$$\Phi_N(t) = \exp\left(-\frac{t^2}{2} \times \left\{\frac{1}{S_N^2}\left(\frac{(N-2)!!}{(N-1)!!}\right)^2 + \frac{S_{N-2}^2}{S_N^2}\right\}\right) = \exp\left(-\frac{t^2}{2}\right). \tag{34}$$

When $\beta \to \infty$ with N fixed, the low-temperature distribution of $d_{2p+1,\beta} = \frac{D_{2p+1,\beta}}{(2p)!!\rho^p S_{2p+1}}$ is thus a $N(0,1)$ Gaussian provided that the asymptotic distributions of $d_{2p-1,\beta}$ and that of $d_{2p,\beta}$ are $N(0,1)$ Gaussians.

As the asymptotic distributions of $d_{2,\beta}$ and $d_{3,\beta}$ are $N(0,1)$ Gaussians (appendices C and D), the low-temperature distribution of the reduced determinant (equation (29)) is, for N fixed and $\beta \to \infty$:

$$P(d_{N,\beta}) = \frac{\exp\left(-d_{N,\beta}^2/2\right)}{\sqrt{2\pi}}. \tag{35}$$

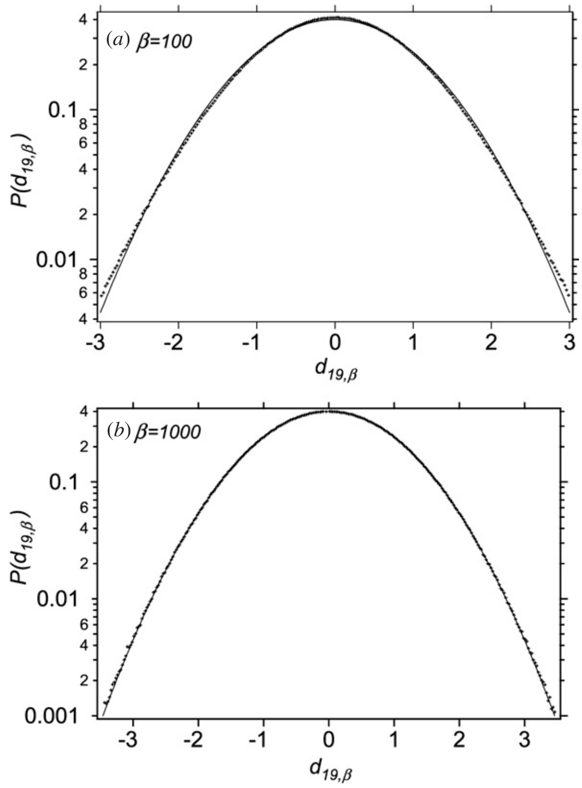

Figure 4. Distribution (points) of the scaled determinant $d_{19,\beta} = D_{19,\beta}/\sqrt{\langle D_{19,\beta}^2\rangle}$ of 19×19 β-Hermite matrices from Monte Carlo simulations with 10^7 matrices (*a*) for $\beta = 100$, (*b*) for $\beta = 1000$ (a vertical log-scale has been used to enhance the resolution in the tails, solid lines = $N(0, 1)$ distributions, $-\frac{\log(2\pi)+d_{19,\beta}^2}{2}$).

The results of numerical simulations, with 10^7 to 10^5 matrices according to N (examples are shown in figures 3 and 4), N varying from 2 to 401 and a typical value $\beta = 1000$, are in excellent agreement with equation (35).

A Gaussian shape $N(0, \sigma^2)$, with $\sigma^2 \neq 1$, is observed to be a good approximation of the determinant distribution for $N = 2p + 1$ even for moderate values of β (figure 4(*a*)). The rate of convergence to the asymptotic distribution remains to be determined as a function of N. It is expected to be slower for even values of N as the odd moments are not zero. A condition on β and N is however discussed below for N even.

As for large k, $\binom{2k}{k}^2/2^{4k} \sim \frac{1}{\pi k}$, the large N values of S_N are obtained from

$$\begin{cases} S_{2p}^2 \approx \dfrac{1}{2}\ln(N) + \dfrac{\gamma + \ln 2 - 2}{2} \\ S_{2p+1}^2 \approx \dfrac{\ln(N)}{\pi} - 0.1512 \end{cases} \tag{36}$$

where γ is the Euler constant. For large N and large ρ, equations (26), (27) and (36) yield a standard-deviation of the determinant which behaves as $\approx(\rho N)^{N/2}\exp\left(-\frac{N}{2}\right)\sqrt{\ln(N)}$. Therefore, a scaling of the elements of a β-H matrix $\propto(\rho N)^{1/2}$, in agreement with section 2.2, leaves a variance of the determinant $\propto\ln(N)$. The latter variance is a direct consequence of the highly correlated fluctuations of the charge positions as analysed via normal modes by Andersen *et al* [43].

4.1.2. The generalized Gumbel distribution of $\ln|d_{N,\beta}|$ *at low temperature.* The asymptotic lognormal distribution discussed in section 1.2, for β fixed and $N \to \infty$, is, in some conditions, compatible with the previous asymptotic normal distribution equation (48). Indeed, when N is large, $N = 2p$, the logarithm of $\left|\frac{D_{2p,\beta}}{\langle D_{2p,\beta}\rangle}\right|$ can be written as (equation (29))

$$\ln\left|\frac{D_{2p,\beta}}{\langle D_{2p,\beta}\rangle}\right| = \ln\left|1 + (-1)^p\sqrt{\frac{\ln N}{\beta}}d_{2p,\beta}\right| \approx (-1)^p\sqrt{\frac{\ln N}{\beta}}d_{2p,\beta} \qquad \textit{if} \quad \beta \gg \ln N. \tag{37}$$

Equation (37) shows that the Gaussian behaviour is recovered at low enough temperatures and with a variance varying as $\ln(N)$. The latter result was proven from the exact determinant distributions for $\beta = 1, 2$ [22]. At infinite temperature, $\beta = 0$, the variance is $\propto N$ [22]. Indeed, it is found, using the values of K_1 and K_2 given by equation (E.4) and the Gumbel distribution discussed below (equation (41)), that

$$\frac{\ln|D_{N,0}| - N(\ln\sigma - (\gamma + \ln 2))/2)}{\pi N^{1/2}/2\sqrt{2}} \tag{38}$$

has an $N(0, 1)$ distribution when $N \to \infty$. In equation (38), γ is the Euler constant and σ is defined in equation (1).

The Gaussian distribution is not only the limit of the lognormal distribution of the determinant of large matrices at low temperature but it is valid whatever N, in particular for small N values for which the latter distribution is irrelevant. As a product of Gaussian variables may be normal in some conditions (see appendix B for the case of independent variables), the overall low-temperature behaviour seems thus to be accounted for in simple terms. However, the observation of a rather atypical distribution for a product of random variables, even correlated, might call for a more general explanation and the correlation structure of the eigenvalues must be looked at more closely as a function of temperature. A generalized Gumbel distribution has the following density ($a > 0$) [38–42]:

$$g_{a,\theta,s}(x) = \frac{a^a|\theta|}{\Gamma(a)}\exp(a\{\theta(x-s) - \exp(\theta(x-s))\}). \tag{39}$$

The parameter θ is found hereafter to be positive. In their investigation of decaying Burgers turbulence, Noullez and Pinton [40] concluded that the parameter a plays the same role as the number of degrees of freedom in the chi-square distribution. There is indeed a direct link between the generalized Gumbel distribution (equation (39)) and the chi-square distribution with $2a$ degrees of freedom, the latter being in fact the distribution of $y = 2a\exp(\theta(x-s))$. Equivalently $z = \sqrt{2a}\exp\left(\frac{\theta(x-s)}{2}\right)$ has a chi distribution with $2a$ degrees of freedom:

$$p_{2a}(z) = \frac{z^{2a-1}}{2^{a-1}\Gamma(a)}\exp\left(-\frac{z^2}{2}\right). \tag{40}$$

The distribution of the logarithm of the absolute value of a standard Gaussian variable, $x = \ln(|N(0, 1)|)$, is thus a generalized Gumbel distribution:

$$g_{1/2,2,0}(x) = \sqrt{\frac{2}{\pi}}\exp\left(x - \frac{1}{2}\exp(2x)\right). \tag{41}$$

Some characteristics of the generalized Gumbel distribution are further discussed in appendix E. The low-temperature distribution $P(x)$ of $x = \ln(|d_{N,\beta}|)$, which is minus the electrostatic

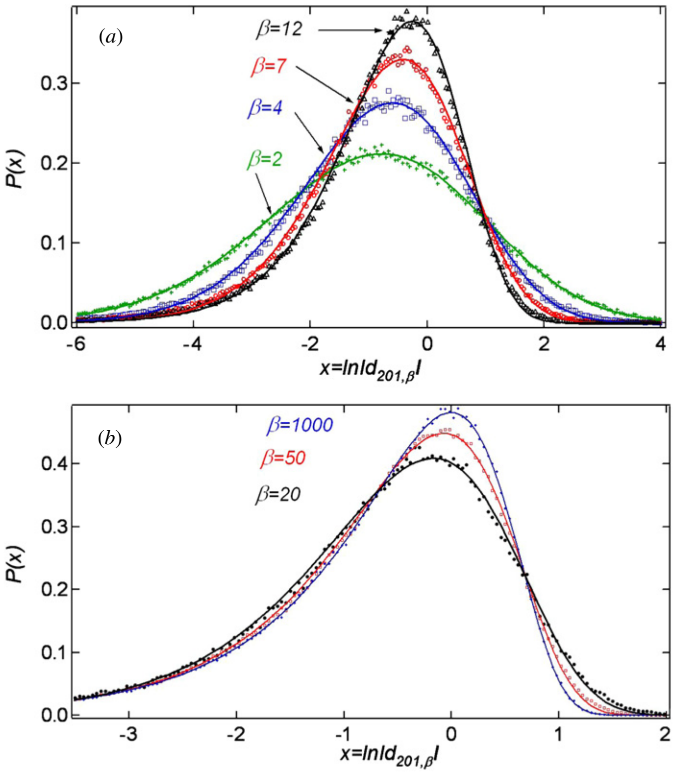

Figure 5. Distribution $P(x)$ of $x = \ln(|d_{201,\beta}|)$ for 201×201 β-Hermite matrices as a function of β from Monte Carlo simulations with 5×10^5 matrices ($d_{201,\beta} = D_{201,\beta}/\sqrt{\langle D^2_{201,\beta}\rangle}$, solid lines = least-squares fits with generalized Gumbel distributions, $g_{a,\theta,s}(x)$ (equation (A.4)) whose parameters are shown in figure 7).

potential at the origin, is then $g_{1/2,2,0}(x)$ whatever N. Figures 5–7 show the evolution of the distribution $P(x)$, for $N = 201$, from a normal distribution to $g_{1/2,2,0}(x)$ and that of some of its characteristics. All distributions of figure 5 were fitted with a generalized Gumbel distribution (equation (39)). Excellent least-squares fits are obtained for $\beta >\sim 20$ (figure 5(*b*)). Fair fits are still obtained for smaller values of β (figure 5(*a*)). The fitted parameters are such that $a(\beta) \sim \frac{1}{\theta(\beta)}$ (figure 7). Figures 6 and 7 show that the parameters change rapidly when β increases up to ~ 50, a value indeed small as compared to $\ln N = 5.303$ and then rather slowly.

The generalized Gumbel distribution (equation (39)) is the asymptotic distribution of the extreme values of sequences correlated over a 'distance' $1/a$ [41]. The problem we are considering here is not an extreme value problem but Bertin and Clusel showed recently [42] that it is possible to obtain the generalized Gumbel distribution as the asymptotic distribution of sums of independent non-identically distributed random variables or of correlated random variables belonging to broad classes which do not satisfy the conditions of validity of the central limit theorem. The parameter a quantifies their correlation structure [42]. For a physical example they consider that of a 1D lattice with L sites and a continuous variable on each site, a is for instance the ratio of the correlation length to the system size L. It is thus relevant to look here at the linear correlations between the ordered eigenvalues ($\lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_N$)

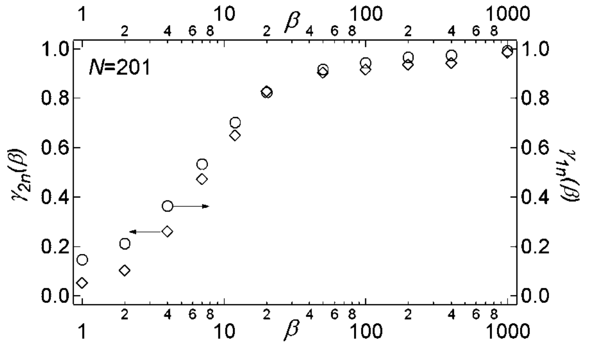

Figure 6. Normalized coefficient of excess $\gamma_{2n}(\beta) = \frac{\gamma_2(\beta)}{\gamma_2(\infty)}$ (empty diamonds) and normalized skewness $\gamma_{1n}(\beta) = \frac{\gamma_1(\beta)}{\gamma_1(\infty)}$ (empty circles) of the distributions $P(x)$ of figure 5 as a function of β ($\gamma_1(\infty) = 4$ and $\gamma_2(\infty) \approx -1.5351$ (appendix E)). Both γ_{1n} and γ_{2n} are zero when $P(x)$ is a Gaussian.

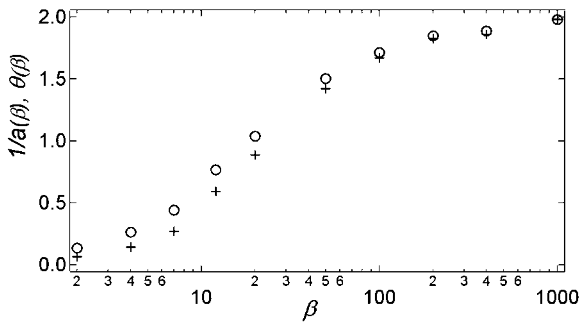

Figure 7. Parameters $1/a(\beta)$ (crosses) and $\theta(\beta)$ (empty circles) of the generalized Gumbel distribution $g_{a,\theta,s}(x)$ used to fit the distributions $P(x)$ of figure 5 ($a(\infty) = 0.5$ and $\theta(\infty) = 2$ (appendix E), θ is zero when $P(x)$ is a Gaussian).

and between $(\ln(|\lambda_1|), \ldots, \ln(|\lambda_N|))$. The linear correlation coefficient between λ_j and λ_k is

$$\begin{cases} \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N \\ \rho_c(j,k) = [\langle \lambda_j \lambda_k \rangle - \langle \lambda_j \rangle \langle \lambda_k \rangle]/(\sigma_j \sigma_k) \\ \sigma_m^2 = \langle \lambda_m^2 \rangle - \langle \lambda_m \rangle^2. \end{cases} \tag{42}$$

The correlation coefficients $\rho_c(j,k)$ do not depend sensitively on β when $\beta \geqslant 1$. Figure 8 exemplifies it by showing the variation of $\rho_c(1,k)$ with $(\lambda_k - \lambda_1)/(\lambda_N - \lambda_1)$ for $\beta = 1$ and for β infinite (figure 8(*a*)). The correlation coefficients between $\ln(|\lambda_1|)$ and $\ln(|\lambda_k|)$ decrease more strongly with k than the $\rho_c(1,k)$'s and their temperature dependences are also very weak. The random variables $x_j (j = 1, \ldots, N)$, which result from an ordering process of N independent realizations of a Gaussian distribution $N(0, \sigma^2)$, appear as more strongly correlated, with a broader reach, than are the previous ordered eigenvalues (figure 8(*b*)). The x_j's have different distributions but the central limit theorem holds obviously for $\ln\left(\prod_{k=1}^{N} |x_k|\right)$ when $N \to \infty$ as it is actually a sum over iid lognormal random variables. This example emphasizes, as do Bertin and Clusel [42], the difficulty to draw a general conclusion about

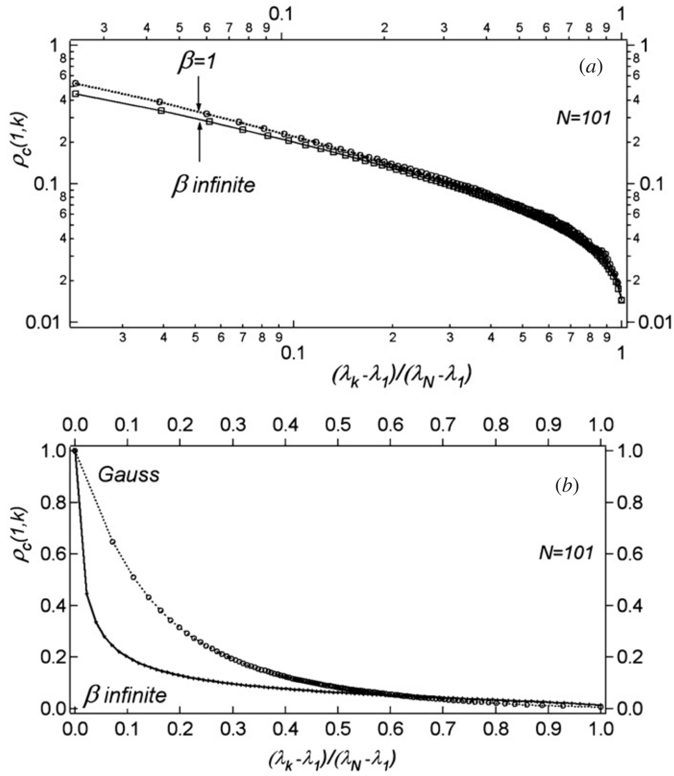

Figure 8. Correlation coefficient $\rho_c(1,k)$ between the smallest element λ_1 and the remaining elements $\lambda_k (k = 2, \ldots, 101)$ of a set of $N = 101$ ordered values as a function of $(\lambda_k - \lambda_1)/(\lambda_N - \lambda_1)$: (*a*) the λ_j's are the eigenvalues of 101 × 101 β-Hermite matrices (solid line and empty squares: β infinite, $\rho_c(1,k)$ calculated from equation (11), dotted line and empty circles: $\beta = 1$, $\rho_c(1,k)$ calculated from Monte Carlo simulations with 10^6 matrices); (*b*) the λ_j's are taken as the ordered elements x_j's $(j = 1, \ldots, N)$ obtained by drawing a random sample of $N = 101$ values from a Gaussian distribution $N(0, \sigma^2)$ (the dotted line is calculated from Monte Carlo simulations with 2×10^6 realizations).

the asymptotic behaviour of a sum of non-independent and non-identically distributed random variables. The sole knowledge of the range of their correlation coefficients may in some cases be of little help. To capture the global evolution, figure 9 presents various distributions $P(|\rho_c|)$ of the absolute value of the $N(N-1)/2$ correlation coefficients $\rho_c(j,k)$, noted ρ_c for brevity. Although the distribution of the correlation coefficient ρ_c between the eigenvalues differs from that between the logarithms of their absolute values, the associated distributions $P(|\rho_c|)$ are essentially indistinguishable. Figure 9(*a*) depicts the weak temperature dependence of the global correlation structure for N fixed, $N = 400$. Finally, figure 9(*b*) shows the evolution with N of $P(|\rho_c|)$ for $\beta = 1$. When N increases, the distribution becomes broader and flatter and moves progressively towards the distribution found for ordered variables from Gaussian realisations.

To conclude, we failed to evidence a significant temperature dependence of the correlation structure for a fixed N. It might have accounted for the progressive evolution of the distribution of $x = \ln(|d_{N,\beta}|)$ from a normal distribution to a generalized Gumbel distribution $g_{1/2,2,0}(x)$, through distributions which are very well approximated too by generalized

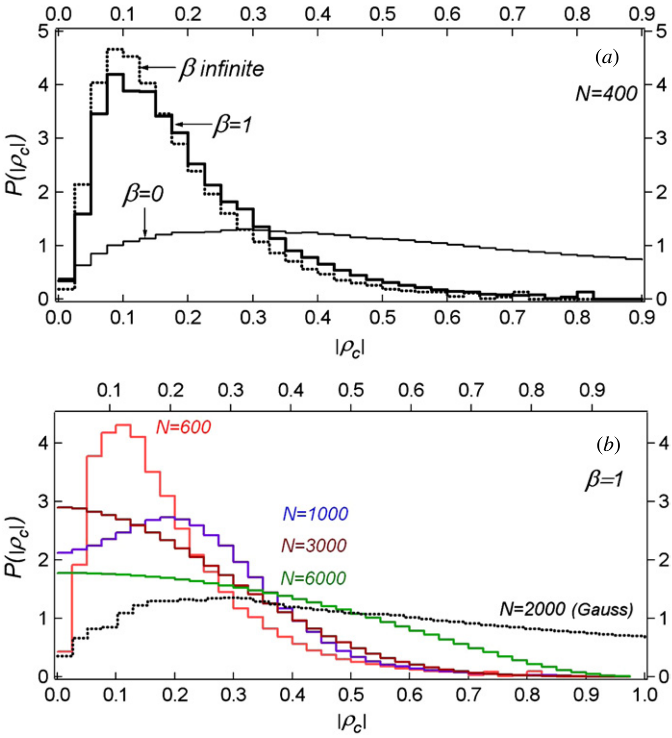

Figure 9. Distribution $P(|\rho_c|)$ of the absolute value $|\rho_c|$ of the correlation coefficient between the ordered eigenvalues of $N \times N$ β-Hermite matrices: (*a*) as a function of β for $N = 400$, (*b*) as a function of N for $\beta = 1$ (distributions $P(|\rho_c|)$ are further shown for ordered values of sets of $N = 2000$ iid Gaussian variables).

Gumbel distributions $g_{a,\theta,s}(x)$, when the temperature decreases. The low-temperature normal distribution of the reduced determinant is at that point concluded to be simply the limiting behaviour of a product of correlated Gaussian variables.

4.2. Fixed-trace β-Hermite ensemble

As the maximum of $\prod_{k=1}^{N} |\lambda_k|$ subject to the constraint $\sum_{k=1}^{N} \lambda_k^2 = 1$ is $N^{-N/2}$ we define first a scaled β-FTH's determinant $D_{N,\beta}$:

$$D_{N,\beta} = N^{N/2} \left[\prod_{k=1}^{N} \lambda_k \right] \tag{43}$$

so that the determinant density of the β-FTHE is

$$P^F(D_{N,\beta}) = 0 \qquad \text{for} \quad |D_{N,\beta}| \geqslant 1 \tag{44}$$

whatever N. The distribution $P^F(D_{2,\beta})$ is given in appendix C. The relation that we obtained between the determinant distributions of the Gaussian ensembles ($\beta = 1, 2$) (equation (15) of [24]) and those of the associated fixed-trace ensembles is still appropriate for the β-Hermite ensemble. Indeed, it is solely based on the chi-square distribution of $\mathrm{tr}\big(\mathbf{H}_{N,\beta}^2\big)$ (section 2.1).

The distribution $P(D_{N,\beta})$ of the determinant of the β-Hermite ensemble is then related to $P^F(D_{N,\beta})$ by [24]

$$P(D_{N,\beta}) = \frac{1}{2^{\frac{N_\rho}{2}-1}\sigma^{N_\rho}\Gamma\left(\frac{N_\rho}{2}\right)} \int_{|D_{N,\beta}|^{\frac{1}{N}}}^{\infty} P^F\left(\frac{D_{N,\beta}}{r^N}\right) r^{N_\rho-N-1} \exp\left(-\frac{r^2}{2\sigma^2}\right) \mathrm{d}r. \tag{45}$$

A relation between the Mellin transforms of the determinant distributions, valid whatever β, can be derived from equation (45) as done for equations (17) and (A.3) of [24] for $\beta = 1$, 2. The moments of the distribution of the determinant of a β-FTH matrix (equation (43)) are then related to those of the determinant of a β-H matrix by

$$\left\langle D_{N,\beta}^k \right\rangle_{\beta\text{-}FTH} = \left\langle D_{N,\beta}^k \right\rangle_{\beta\text{-}H} \times \left(\frac{N}{2}\right)^{\frac{Nk}{2}} \times \frac{\Gamma(N_\rho/2)}{\Gamma(N_\rho/2 + Nk/2)}. \tag{46}$$

As the β-HE tends to the β-FTHE for large βN^2 (section 3.2), the determinant of a β-FTH matrix should behave at low temperature as that a β-H matrix. For simplicity, we show it only for N odd, $N = 2p + 1$. Indeed, $\left\langle D_{2p+1,\beta}^{2k} \right\rangle_{\beta-H} = (2k-1)!!\sigma_{\beta-H,\infty}^{2k}$ for large β, where $\sigma_{\beta-H,\infty}^2 = [(N-1)!!]^2 S_N^2 \rho^{N-1}$ is the asymptotic variance obtained from equation (29). Then:

$$\left\langle D_{N,\beta}^{2k} \right\rangle_{\beta\text{-}FTH} = (2k-1)!! \left(\frac{\sigma_{\beta-H,\infty}}{(\rho(N-1))^{N/2}}\right)^{2k}. \tag{47}$$

Equation (47) yields a reduced determinant whose asymptotic distribution is a $N(0, 1)$ Gaussian:

$$d_{N,\beta} = \frac{(N(N-1))^{N/2}\sqrt{\rho}}{(N-1)!!S_N} \left[\prod_{k=1}^{N} \lambda_k\right]. \tag{48}$$

Results of numerical simulations are in excellent agreement with equation (48). A similar calculation can be performed for N even ($N \geqslant 4$) with a nonzero scaled average $\langle d_{2p,\beta}\rangle = (-1)^p \frac{\sqrt{\rho}}{S_{2p}}$. As expected, a $\ln(N)$ variance is obtained from equation (48) for large N.

5. Conclusions

To conclude, the β-Hermite ensemble of tridiagonal $N \times N$ random matrices of Dumitriu and Edelman [11] is a continuum of ensembles which facilitates the exploration of the spectral properties in the whole temperature range among others thanks to the numerical efficiency it brings. A fixed-trace β-Hermite ensemble is defined from the β-Hermite ensemble and is used to extend spherical ensembles of classical symmetries to β-spherical ensembles.

When $\beta \to \infty$ for a fixed value of N, the low-temperature distribution of the unscaled determinant of a random β-Hermite matrix and that of a random β-fixed trace Hermite matrix are Gaussians whose variances are determined. The low temperature Gaussian distribution is actually that of the product of eigenvalues which are correlated normal random variables whose multivariate distribution is explicitly known [12]. The asymptotic normal distribution of the determinant is derived here by a simple inductive reasoning which is not based on the latter fact. Otherwise stated, the reduced potential at the origin, $V_0 = -\ln|d_{N,\beta}|$, has a generalized Gumbel distribution, $g_{1/2,-2,0}(V_0)$, at low temperature for a fixed value of N. For large N and large β, a $\ln(N)$ variance of the distribution of the determinant of a scaled β-H matrix results from the strongly correlated fluctuations of eigenvalues around their equilibrium positions. The normal distribution holds whatever N at low temperature. For large N, it is the form taken by the classical lognormal distribution of the determinant when $\beta \gg \ln N$. The low- temperature determinant distribution is expected to hold for other ensembles whose eigenvalues show strongly localized Gaussian fluctuations around the zeros of the orthogonal polynomials associated with the probability measures which define these ensembles.

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Appendix A. Distributions of the elements of an $N \times N$ β-FTH matrix

The marginal distributions of a diagonal element F_{kk} and of an off-diagonal element $F_{k,k+1}$ of a β-FTH matrix are obtained from equation (17):

$$\begin{cases} P(F_{kk}) = \dfrac{\Gamma(N_\rho/2)}{\sqrt{\pi}\Gamma((N_p - 1)/2)} \times \left(1 - F_{kk}^2\right)^{(N_\rho-3)/2} \\ -1 \leqslant F_{kk} \leqslant 1, \quad k = 1, \ldots, N \end{cases} \tag{A.1}$$

with a variance $\left\langle F_{kk}^2 \right\rangle = \frac{1}{N_\rho}$ and

$$\begin{cases} P(F_{k,k+1}) = \dfrac{2\Gamma(N_\rho/2)}{\Gamma(k\beta/2)\Gamma((N_\rho - k\beta)/2)} \times F_{k,k+1}^{k\beta-1} \times \left(1 - F_{k,k+1}^2\right)^{(N_\rho-k\beta-2)/2} \\ 0 \leqslant F_{k,k+1} \leqslant 1, \quad k = 1, \ldots, N-1 \end{cases} \tag{A.2}$$

with $\left\langle F_{k,k+1} \right\rangle = \frac{\Gamma(N_\rho/2)}{\Gamma((N_\rho+1)/2)} \times \frac{\Gamma((k\beta+1)/2)}{\Gamma(k\beta/2)}$ and $\left\langle F_{k,k+1}^2 \right\rangle = \frac{k\beta}{N_\rho}$.

The amalgamation property (section 3) further shows that the distribution of $F_{kk}(k = 1, \ldots, m)$, for integer values of β, is nothing else than the marginal distribution of any component of a unit vector $\mathbf{U}_{N_\rho}$ uniformly distributed on the surface of the unit sphere in R^{N_ρ} (appendix A of [32, 49]). Indeed, the distribution of the squares of the components of $\mathbf{U}_{N_\rho}$ is a Dirichlet distribution whose parameters ν_k are all equal to 1 [49, p 20] as are those of the diagonal elements of $\mathbf{F}_{N,\beta}$ (equation (15)). The joint distribution of the diagonal elements of $\mathbf{F}_{N,\beta}$ is obtained from equation (17):

$$p(F_{11}, \ldots, F_{NN}) = \frac{\Gamma(N_\rho/2)}{\pi^{N/2}\Gamma((N_\rho - N)/2)} \times \left(1 - \sum_{k=1}^{N} F_{kk}^2\right)^{(N_\rho-N-2)/2}. \tag{A.3}$$

The distribution of the trace, $\mathrm{tr}(\mathbf{F}_{N,\beta})$, is calculated by performing first a change of variables from $\mathbf{D}_N = (F_{11}, \ldots, F_{NN})$ to $\mathbf{D}'_N = (F'_{11}, \ldots, F'_{NN})$, with $\mathbf{D}'_N = \mathbf{O}\mathbf{D}_N$ where $\mathbf{O}$ is an orthogonal matrix whose first line has all its elements equal to $1/\sqrt{N}$, for instance an Helmert matrix [51], so that $F'_{11} = \sum_{k=1}^{N} F_{kk}/\sqrt{N}$. The distribution of $\mathbf{D}'_N$ is still given by equation (A.3) $\left(\sum_{k=1}^{N} F_{kk}^2 = \sum_{k=1}^{N} F_{kk}'^2\right)$ and the distribution of $x = \mathrm{tr}(\mathbf{F}_{N,\beta})/\sqrt{N}$ is then nothing else that the distribution of F'_{11} given by equation (A.1). The distribution of the scaled trace $t = \mathrm{tr}(\mathbf{F}_{N,\beta}) \times \sqrt{\frac{\beta N}{2}}$ is then

$$p(t) = \sqrt{\frac{2}{\pi\beta}} \times \frac{1}{N} \times \frac{\Gamma(N_\rho/2)}{\Gamma((N_\rho - 1)/2)} \times \left(1 - \frac{2t^2}{\beta N^2}\right)^{(N_\rho-3)/2}. \tag{A.4}$$

Distribution equation (A.4) converges rapidly to an $N(0, 1)$ Gaussian distribution when $\beta N^2 \to \infty$.

Appendix B. The distribution of a product of *N* independent non-central Gaussian variables

We consider the product D_N of N independent Gaussians Y_k:

$$\begin{cases} Y_k = m_k + X_k = N(m_k, \sigma^2) \\ D_N = \prod_{k=1}^{N} (m_k + X_k) \\ m_k \neq 0, \quad k = 1, \ldots, N \end{cases} \tag{B.1}$$

where the X_k's are iid $N(0, \sigma^2)$ Gaussians. The three first centred moments are then

$$\begin{cases} \dfrac{\langle D_N^2 \rangle_c}{\langle D_N \rangle^2} = \prod_{k=1}^{N} (1 + z_k) - 1 \qquad \dfrac{\langle D_N^3 \rangle_c}{\langle D_N \rangle^3} = \prod_{k=1}^{N} (1 + 3z_k) - 3 \prod_{k=1}^{N} (1 + z_k) + 2 \\ \dfrac{\langle D_N^4 \rangle_c}{\langle D_N \rangle^4} = \prod_{k=1}^{N} \left(1 + 6z_k + 3z_k^2\right) - 4 \prod_{k=1}^{N} (1 + 3z_k) + 6 \prod_{k=1}^{N} (1 + z_k) - 3 \end{cases} \tag{B.2}$$

with $z_k = \frac{\sigma^2}{m_k^2}$. If the distribution of D_N (equation (B.1)) is dominated by the linear term of its expansion, then this distribution is essentially Gaussian, being well approximated by a linear combination of Gaussian variables. For that purpose, we impose that the leading terms in the expansions of the first moments determine their values. The latter terms are

$$\frac{\langle D_N^2 \rangle_c}{\langle D_N \rangle^2} = \sigma_D^2 + \cdots \qquad \frac{\langle D_N^3 \rangle_c}{\langle D_N \rangle^3} = 3\sigma^4 S_2 + \cdots \qquad \frac{\langle D_N^4 \rangle_c}{\langle D_N \rangle^4} = 3\sigma_D^4 + \cdots \tag{B.3}$$

with

$$S_1 = \sum_{i=1}^{N} \frac{1}{m_i^2} \qquad S_2 = \sum_{i \neq j=1}^{N} \frac{1}{m_i^2 m_j^2} \qquad \sigma_D^2 = \sigma^2 S_1. \tag{B.4}$$

The second term in the expansion of the variance is $\sigma^4 S_2/2$. The first term of the fourth moment is that expected for a Gaussian $N(0, \sigma_D^2)$. We determine σ and ε by the conditions that both the absolute value of the scaled third moment, $\left| \frac{\langle D_N^3 \rangle_c}{\sigma_D^3 \langle D_N \rangle^3} \right| = \varepsilon$ and the second term in the expansion of $\frac{\langle D_N^2 \rangle_c}{\langle D_N \rangle^2} \Big/ \sigma_D^2$ are negligible as compared to 1, namely $\varepsilon \ll (18S_2)^{1/2}/S_1$ and $\sigma = \varepsilon S_2^{3/2} \big/ 3S_1$. The ratio $\sqrt{S_2}/S_1$ is less than 1 and can be very small. In the most general conditions, the properly scaled product has most often a lognormal distribution when $N \to \infty$.

Appendix C. Some characteristics of 2 × 2 β-H and β-FTH matrices

C.1. β-HE

Whatever β, the exact eigenvalue density is easily obtained to be

$$p_{\beta-H,2}(\lambda) = \frac{\exp(-\lambda^2/2\sigma^2)}{2^\rho \sigma \sqrt{2\pi}} \Phi \left(-\rho, \frac{1}{2}; -\frac{\lambda^2}{2\sigma^2} \right) \tag{C.1}$$

where $\Phi(a, b; x)$ is a degenerate hypergeometric function [47]. For large β, the density is then very well approximated by the sum of two correlated Gaussians in agreement with the results of Dumitriu and Edelman (section 2.2).

The distribution of the determinant $D_{2,\beta}$ is asymmetric with a mean equal to $-\rho$ (figure 3(*a*)). The exact values of the central moments about the mean $\langle D_{2,\beta}^n\rangle_c$, $n = 2, \ldots, 7$ are

$$\begin{cases} \langle D_{2,\beta}^2\rangle_c = 1+\rho & \langle D_{2,\beta}^3\rangle_c = -2\rho \\ \langle D_{2,\beta}^4\rangle_c = 9+12\rho+3\rho^2 & \langle D_{2,\beta}^5\rangle_c = -44\rho-20\rho^2 \\ \langle D_{2,\beta}^6\rangle_c = 225+345\rho+175\rho^2+15\rho^3 & \langle D_{2,\beta}^7\rangle_c = -1854\rho-1344\rho^2-210\rho^3. \end{cases}$$

The reduced determinant is (equation (40))

$$d = \frac{D_{2,\beta}+\rho}{\rho^{1/2}} = \frac{x_1x_2-(y/2-\rho)}{\rho^{1/2}} \tag{C.2}$$

where x_1, x_2, y are independent and have respectively $N(0,1)$, $N(0,1)$ Gaussian distributions and a chi-square distribution with β degrees of freedom. The characteristic function of d is then

$$\Phi_{2,\beta}(t) = \langle \exp(\mathrm{i}td)\rangle = \frac{\exp(\mathrm{i}t\rho^{1/2})}{(1+\mathrm{i}t\rho^{-1/2})^\rho} \times \frac{1}{\sqrt{1+\rho^{-1}t^2}}. \tag{C.3}$$

The large ρ expansion of $(1+\mathrm{i}t\rho^{-1/2})^\rho$ shows finally that $\Phi_{2,\beta}(t)$ tends to $\Phi_{2,\infty}(t) = \exp\left(-\frac{t^2}{2}\right)$ when $\beta \to \infty$. The asymptotic distribution of $\frac{D_{2,\beta}+\rho}{\sqrt{\rho}}$ is thus a $N(0,1)$ Gaussian.

C.2. β-FTHE

The determinant density of the β-FTHE, $P^F(D_{2,\beta})$, is deduced from that of the trace $z = F_{11}+F_{22}$ obtained from equation (A.4) with $N = 2$. From $D_{2,\beta} = 2\lambda_1\lambda_2 = z^2-1$ it follows that

$$P^F(D_{2,\beta}) = \frac{\Gamma(1+\rho)}{2^\rho\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\rho\right)} \times (1+D_{2,\beta})^{-1/2}(1-D_{2,\beta})^{\rho-1/2} \tag{C.4}$$

in agreement with our previous results for $\beta = 1,2$ (equation (16) of [24]). The first centred moments are

$$\begin{cases} \langle D_{2,\beta}\rangle_F = -\dfrac{\rho}{1+\rho} \qquad \langle D_{2,\beta}^2\rangle_{F,c} = \dfrac{1+2\rho}{(1+\rho)^2(2+\rho)} \\ \langle D_{2,\beta}^3\rangle_{F,c} = \dfrac{4\rho(1+2\rho)}{(1+\rho)^3(2+\rho)(3+\rho)}. \end{cases} \tag{C.5}$$

Defining, for large ρ, $x^2 = \rho(D_{2,\beta}+1)$, the distribution of x obtained from equation (C.1) tends to a chi distribution with one degree of freedom:

$$P^F(x) = \sqrt{\frac{2}{\pi}}\exp(-x^2/2) \qquad (x > 0). \tag{C.6}$$

Appendix D. The even moments of $D_{3,\beta}$ for $\beta \to \infty$ ($\sigma = 1$)

From the definition of the β-HE (equation (2)) we get

$$\langle D_{3,\beta}^{2p}\rangle = \left\langle \left(N_1N_2N_3 - \frac{N_3}{2}\chi_\beta^2 - \frac{N_1}{2}\chi_{2\beta}^2\right)^{2p}\right\rangle \tag{D.1}$$

where the five random variables in the right member are independent and N_1, N_2, N_3 are iid $N(0,1)$. The highest degree ρ^{2p} term is simply calculated to be

$$\langle D_{3,\beta}^{2p}\rangle = (5\rho^2)^p(2p-1)!! \tag{D.2}$$

which shows that the asymptotic distribution of $D_{3,\beta}$ is an $N(0, 5\rho^2)$ Gaussian ($\sigma = 1$ in equation (1)). The exact values of $\langle D_{3,\beta}^{2p}\rangle$, $p = 1,2,3$, are

$$\begin{cases} \langle D_{3,\beta}^2\rangle = 1 + 3\rho + 5\rho^2 \\ \langle D_{3,\beta}^4\rangle = 27 + 108\rho + 267\rho^2 + 198\rho^3 + 75\rho^4 \\ \langle D_{3,\beta}^6\rangle = 3375 + 15\,525\rho + 45\,750\rho^2 + 51\,975\rho^3 + 35\,400\rho^4 + 11\,475\rho^5 + 1875\rho^6. \end{cases}$$

Appendix E. The generalized Gumbel distribution

The generalized Gumbel distribution considered below is ($a > 0$)

$$g_{a,\theta,s}(x) = \frac{a^a|\theta|}{\Gamma(a)} \exp(a\{\theta(x-s) - \exp(\theta(x-s))\}). \tag{E.1}$$

The cumulants K_n are thus deduced from the successive derivatives of the logarithm of the characteristic function $\Phi(t) = \langle e^{itx}\rangle = \frac{e^{its}\Gamma(a+it/\theta)}{\Gamma(a)a^{it/\theta}}$ [52] at $t = 0$:

$$\begin{cases} K_1 = \langle x\rangle = s + (\psi(a) - \ln(a))/\theta \\ K_2 = \langle (x-K_1)^2\rangle = \psi^{(1)}(a)/\theta^2 \\ K_3 = \langle (x-K_1)^3\rangle = \psi^{(2)}(a)/\theta^3 \\ K_4 = \langle (x-K_1)^4\rangle - 3K_2^2 = \psi^{(3)}(a)/\theta^4 \\ K_n = \psi^{(n-1)}(a)/\theta^n \qquad n > 1 \end{cases} \tag{E.2}$$

where $\psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}} \ln\Gamma(x)$ is a polygamma function and [48]

$$\psi^{(n)}(a) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(a+k)^{n+1}}. \tag{E.3}$$

The coefficient of skewness $\gamma_1 = \frac{K_3}{K_2^{3/2}}$ is a measure of the asymmetry of the distribution with a tendency to tail to the left when γ_1 is negative. The coefficient of excess $\gamma_2 = \frac{K_4}{K_2^2}$ is a measure of shape. When it is positive, the distribution has longer tails than the normal distribution. Both coefficients are zero for a Gaussian distribution. For $a = 1/2$, $\theta = 2$, $s = 0$ the previous cumulants and coefficients become

$$\begin{cases} K_1 = -\dfrac{(\gamma + \ln 2)}{2} = -0.635\,181\ldots & K_2 = \dfrac{\pi^2}{8} = 1.233\,700\ldots \\ K_3 = -\dfrac{7\zeta(3)}{4} = -2.103\,599\ldots & K_4 = \dfrac{\pi^4}{16} = 6.088\,068\ldots \\ K_n = (-1)^n(n-1)!\left(\dfrac{2^n-1}{2^n}\right)\zeta(n) & n > 1 \\ \gamma_1 = -\dfrac{28\sqrt{2}\zeta(3)}{\pi^3} = -1.535\,141\ldots & \gamma_2 = 4. \end{cases} \tag{E.4}$$

The latter distribution is shown in figure 5(*b*) ($\beta = 1000$).

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